On the flow of a fluid–particle mixture between two rotating cylinders, using the theory of interacting continua

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Abstract

Mixture theory is used to develop a model for a flowing mixture of solid particulates and a fluid. Equations describing the flow of a two-component mixture consisting of a Newtonian fluid and a granular solid are derived. These relatively general equations are then reduced to a system of coupled ordinary differential equations describing Couette flow between concentric rotating cylinders. The resulting boundary value problem is solved numerically and representative results are presented. Published by Elsevier Science Ltd.

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1. Introduction

In the last several decades, there has been a tremendous increase in fundamental studies in the field of 'multiphase flows' in terms of studying (i) issues of importance such as design of nuclear reactors, silos, fluidized beds (and combustion related phenomena), etc., or (ii) trying to understand the flow of avalanches or debris, and other natural phenomena, such as flow through porous media. Historically, the emphasis of research has been on experimental techniques where certain correlations would be obtained with limited ranges of applicability and use [1].

The modeling of multicomponent flows has become a subject of considerable interest. The flow of mixtures consisting of particles entrained in a fluid is relevant to a variety of applications such as fluidized beds and pneumatic transport of solid particles. Two theories used to model these types of fluid–solid flows are averaging and mixture theory (theory of interacting continua). In the averaging approach (cf. Refs. [2–4]) point-wise equations of motion, valid for a single fluid or a single particle, are modified to account for the presence of the other components and the interactions between components. These equations are then averaged over time or some suitable volume.

The details of this technique and other relevant issues can be found in books by Hestroni [5], Meyer [6], Papanicolaou [7], Soo [8,9], Marcus et al. [10], Kaviany [11], Fan and Zhu [12], etc.

Mixture theory, which traces its origins to the work of Fick [13], was first presented within the framework of continuum mechanics by Truesdell [14]. In this approach, the equations and principles

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Nomenclature

$\mathbf{a}$	acceleration vector
$\mathbf{a}_{vm}$	frame-indifferent relative acceleration
A_i	interaction coefficients, $i = 1, \dots, 5$
$\mathbf{b}$	body force vector
$\hat{B}_i$	dimensionless β's
B_i	constant part of $\hat{B}$'s
$\mathscr{B}$	dimensionless parameter
C_i	dimensionless A's
$\mathbf{D}$	stretching tensor
$\mathscr{D}_1$	dimensionless parameter
$\mathscr{D}_2$	dimensionless parameter
$\mathbf{f}_{\mathrm{I}}$	interaction force vector
Fr	Froude number
$\mathscr{F}$	volume fraction dependence of drag
g	acceleration of gravity
$\mathbf{I}$	identity tensor
$\mathbf{L}$	gradient of a velocity vector
$\mathscr{L}$	dimensionless parameter
N	average volume fraction
p	fluid pressure
P	dimensionless fluid pressure
Q	volumetric flow rate of mixture
Q_{m}	mass flow rate of mixture
Re	Reynolds number
$\mathbf{T}$	stress tensor
U	solid velocity
$\mathbf{v}$	velocity vector
V	fluid velocity
$\mathbf{V}$	dimensionless velocity vector
$\mathbf{W}$	spin tensor
x	direction of flow between the plates
$\mathbf{x}$	position vector
X	dimensionless x
$\mathbf{X}$	dimensionless position vector
y	direction normal to plates
Y	dimensionless y

Greek letters

β_i	granular solid coefficients, $i = 0, \dots, 4$
λ_{f}	second coefficient of viscosity, fluid
Λ	dimensionless λ_{f}
μ	fluid viscosity (first coefficient of viscosity)
ν	volume fraction of the solid
ρ	density
τ	dimensionless time
ϕ	volume fraction of fluid

Subscripts

1, f	referring to the fluid phase
2, s	referring to the solid phase
m	referring to mixture

Superscripts

T	transpose
$*$	dimensionless quantity

Other symbols

div	divergence operator
grad	gradient operator
tr	trace of a tensor
∇	gradient symbol
$\otimes$	outer (dyadic) product
$\cdot$	inner (scalar) product

of the mechanics of a single continuum are generalized to include any number of superimposed continua. The fundamental assumption of the theory is that at any instant of time, every point in space is occupied by one particle from each constituent, in a homogenized sense. The historical development and details of mixture theory are given in the review articles by Atkin and Craine [15], Bedford and Drumheller [16], Bowen [17], and several appendices in the recent edition of *Rational Thermodynamics* [18]. El-Kaissy [19], Homsy et al. [20], Ahmadi [21,22], Passman and Nunziato [23], Passman et al. [24], and Massoudi [25,26] have used such an approach for modeling fluid–solid systems. Both averaging and mixture theory require constitutive relations for the stress tensor of each component of the mixture and for momentum exchange between the components. The details of mixture theory can be found in the books by Dobran [27], and Rajagopal and Tao [28].

This paper is concerned with the use of mixture theory to model solid particles in a fluid. Many previous models, whether using averaging or mixture theory, have relied upon an assumption that the solid particles behave as a linearly viscous fluid [2,19,29] with a viscosity μ_s and a pressure field p_s. The meaning of p_s in this context is unclear and it leads to an indeterminancy in the governing equations [25]. This indeterminancy is overcome by assuming a relationship between the solid pressure p_s and the fluid pressure p_f in order to reduce the number of unknowns. A typical assumption is that $p_s = p_f$. This assumption may be justified if the mixture is composed of materials like water and steam, but is inappropriate when one component is a granular solid. Some of the two-fluid models are also inconsistent in that they fail to reduce to the appropriate single constituent model in the two extreme limits.

Recently, Massoudi [26] advocated modeling the stress in the solid constituent of the mixture by a constitutive expression appropriate for flowing granular solids. The model described herein incorporates constitutive equations for a linearly viscous fluid and a flowing granular solid. The model is formulated in a manner such that the mathematical equations reduce to those describing a linearly viscous fluid when the solid volume fraction goes to zero and to those describing a flowing granular solid when the fluid volume fraction goes to zero. Note that physically the volume fraction of the solid particles can never be 1.

Passman et al. [24] use mixture theory to study problems similar to the one analyzed here. Our approach differs significantly from theirs in that our constitutive expressions for the solid stress tensor and the interaction forces are different. Moreover, we do not need to introduce concepts like that of equilibrated forces and stresses that introduce an additional balance equation. We find that the usual balance laws of continuum mechanics are sufficient to study the problem under consideration.

The purpose of this work is to study the effects of interactions and granular material properties on the fluid velocity, solid velocity, and volume fraction profiles determined by solving a specific boundary value problem. First, we review briefly the basic principles of mixture theory and discuss constitutive equations for the mixture components and for the interactions between components. We then derive and non-dimensionalize the general equations governing a flowing mixture of a linearly viscous fluid and a granular solid. The ordinary differential equations describing steady flow of the mixture between rotating cylinders are then presented along with appropriate boundary conditions. Finally, numerically calculated velocity and volume fraction profiles are presented and discussed.

The numerical results are interpreted in terms of dimensionless parameters depending upon the granular material properties, the fluid density and viscosity, the drag coefficient, and the lift coefficients. The influence of the volume fraction on the velocity profiles is to cause a relative decrease in fluid velocity in areas of higher solid concentration and an increase in areas of lower solid concentration. The effect on the solid is, appropriately, just the opposite; the solid moves faster in areas of higher concentration and slower when the concentration is lower. Varying the magnitude of the drag also yields consistent results in that fluid velocities decrease and solid velocities increase with increasing drag coefficient. Lift effects represent the influence of the velocity profiles on the distribution of solid in the flow. The change in the distribution of the solid due to lift effects results in an increase in fluid velocity near the plates and a decrease in fluid velocity where solid concentration has increased (in the center of the flow).

2. Preliminaries

A brief review of the notation and basic equations of mixture theory is presented in this section. Detailed information, including an account of the historical development, is available in the review articles by Atkin and Craine [15] and Bowen [17].

The mixture of fluid and solid is considered to be a purely mechanical system. That is, thermal effects and chemical reactions are ignored. The fluid in the mixture will be represented by S_1 and the granular solid by S_2. At each instant of time, t, it is assumed that each point in space is occupied by particles belonging to both S_1 and S_2. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ denote the positions of particles of S_1 and S_2 in the reference configuration. The motion of the constituents is represented by the mappings

$$\mathbf{x}_1 = \boldsymbol{\chi}_1(\mathbf{X}_1, t) \quad \text{and} \quad \mathbf{x}_2 = \boldsymbol{\chi}_2(\mathbf{X}_2, t), \tag{1}$$

where the subscripts 1 and 2 refer to the fluid and granular solid, respectively. These motions are assumed to be one-to-one, continuous, and invertible. The kinematical quantities associated with these motions are:

$$\mathbf{v}_1 = \frac{\mathrm{D}_1 \boldsymbol{\chi}_1}{\mathrm{D}t}, \quad \mathbf{v}_2 = \frac{\mathrm{D}_2 \boldsymbol{\chi}_2}{\mathrm{D}t}, \tag{2}$$

$$\mathbf{a}_1 = \frac{\mathrm{D}_1 \mathbf{v}_1}{\mathrm{D}t}, \quad \mathbf{a}_2 = \frac{\mathrm{D}_2 \mathbf{v}_2}{\mathrm{D}t}, \tag{3}$$

$$\mathbf{L}_1 = \frac{\partial \mathbf{v}_1}{\partial \mathbf{x}_1}, \quad \mathbf{L}_2 = \frac{\partial \mathbf{v}_2}{\partial \mathbf{x}_2}, \tag{4}$$

$$\mathbf{D}_1 = \tfrac{1}{2}(\mathbf{L}_1 + \mathbf{L}_1^{\mathrm{T}}), \quad \mathbf{D}_2 = \tfrac{1}{2}(\mathbf{L}_2 + \mathbf{L}_2^{\mathrm{T}}), \tag{5}$$

$$\mathbf{W}_1 = \tfrac{1}{2}(\mathbf{L}_1 - \mathbf{L}_1^{\mathrm{T}}), \quad \mathbf{W}_2 = \tfrac{1}{2}(\mathbf{L}_2 - \mathbf{L}_2^{\mathrm{T}}), \tag{6}$$

where $\mathbf{v}$ denotes velocity, $\mathbf{a}$ acceleration, $\mathbf{L}$ is the velocity gradient, $\mathbf{D}$ the stretching tensor, and $\mathbf{W}$ the spin tensor. $\mathrm{D}_1/\mathrm{D}t$ denotes differentiation with respect to t, holding $\mathbf{X}_1$ fixed, $\mathrm{D}_2/\mathrm{D}t$ denotes the same operation holding $\mathbf{X}_2$ fixed.

Also, ρ_1 and ρ_2 are the bulk densities (i.e., the mass of the component per unit mixture volume) of the mixture components given by

$$\rho_1 = \phi \rho_{\mathrm{f}}, \quad \rho_2 = v \rho_{\mathrm{s}}, \tag{7}$$

where ρ_{f} is the density of the pure fluid, ρ_{s} is the density of the solid grains, and v is the volume fraction of the solid component and ϕ is the volume fraction of the fluid. For a saturated mixture $\phi = 1 - v$. The mixture density, ρ_{m} is given by

$$\rho_{\mathrm{m}} = \rho_1 + \rho_2 \tag{8}$$

and the mean velocity $\mathbf{v}$ of the mixture is defined by

$$\rho_{\mathrm{m}} \mathbf{v} = \rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2. \tag{9}$$

Conservation of mass: Assuming no interconversion of mass between the two constituents, conservation of mass for the fluid and solid is

$$\frac{\partial \rho_1}{\partial t} + \operatorname{div}(\rho_1 \mathbf{v}_1) = 0 \tag{10}$$

and

$$\frac{\partial \rho_2}{\partial t} + \operatorname{div}(\rho_2 \mathbf{v}_2) = 0. \tag{11}$$

Conservation of linear momentum: Let $\mathbf{T}_1$ and $\mathbf{T}_2$ denote the partial stress tensors of the fluid S_1 and the solid S_2, respectively. Then the balance of linear momentum for the fluid and solid are given by

$$\rho_1 \frac{\mathrm{D}_1 \mathbf{v}_1}{\mathrm{D}t} = \operatorname{div} \mathbf{T}_1 + \rho_1 \mathbf{b}_1 + \mathbf{f}_{\mathrm{I}}, \tag{12}$$

$$\rho_2 \frac{\mathrm{D}_2 \mathbf{v}_2}{\mathrm{D}t} = \operatorname{div} \mathbf{T}_2 + \rho_2 \mathbf{b}_2 - \mathbf{f}_{\mathrm{I}}, \tag{13}$$

where $\mathbf{b}$ represents the body force, and $\mathbf{f}_{\mathrm{I}}$ represents the mechanical interaction (local exchange of momentum) between the components.

Conservation of moment of momentum: This principle implies that

$$\mathbf{T}_1 + \mathbf{T}_2 = \mathbf{T}_1^{\mathrm{T}} + \mathbf{T}_2^{\mathrm{T}}. \tag{14}$$

The partial stresses need not be symmetric, however.

3. Constitutive equations

Depending on what type of mixture we have, several interesting modeling issues can arise. If there is a fluid which is diffusing through a solid layer, there are a class of problems in rubber elasticity, porous media, etc., that have been studied (cf. Ref. [28]). If, on the other hand, we restrict ourselves to the particulate matter, i.e., an assembly of solid particles infused with a fluid, where the particles or granules can now move freely and interact with the other phase, then the issues of interest become, for example, the modeling of the stress tensors, or the interaction forces. In either case the issue of selecting appropriate and physically meaningful boundary conditions has limited solving practical boundary value problems.

We assume that the fluid and solid components are dense enough so that the theory of interacting continua may be applied. Based on our knowledge of modeling in the theory of granular materials and a linearly viscous fluid, it would be natural to assume all the constitutive functions depend on (cf. Ref. [30]):

$$\rho_1, \rho_2, \nabla\rho_1, \nabla\rho_2, \nabla\nabla\rho_1, \nabla\nabla\rho_2, \mathbf{v}_1 - \mathbf{v}_2, \mathbf{D}_1, \mathbf{D}_2. \tag{15}$$

Then, using methods that are now standard in continuum mechanics (cf. Ref. [15,31]), we can obtain restrictions and forms for such constitutive expressions. Here, we use an alternative approach which is to postulate the constitutive expressions by simply generalizing the structure of the constitutive relations from a single constituent theory. In general, the constitutive expressions for $\mathbf{T}_\mathrm{f}$ and $\mathbf{T}_\mathrm{s}$ depend on the kinematical quantities associated with both the constituents. However, we assume that $\mathbf{T}_\mathrm{s}$ and $\mathbf{T}_\mathrm{f}$ depend only on the kinematical quantities associated with the solid and fluid, respectively. This assumption is sometimes called "the principle of phase separation" (cf. Ref. [4]) and was first put forward by Adkins [32,33].

In the majority of fluid–solid mixtures, the fluid is either a gas or water. Therefore, it is appropriate to assume that the fluid behaves as a linearly viscous fluid, whose constitutive equation is

$$\mathbf{T}_\mathrm{f} = [-p(\rho_1) + \lambda_\mathrm{f}(\rho_1)\mathrm{tr}\,\mathbf{D}_1]\mathbf{I} + 2\mu_\mathrm{f}(\rho_1)\mathbf{D}_1, \tag{16}$$

where p is the fluid pressure, λ_f and μ_f are the viscosities, $\mathbf{D}_1$ is the stretching tensor for the fluid defined in Eq. (5), and $\mathbf{I}$ is the identity tensor. If the fluid is incompressible, then p is one of the unknown quantities in the problem that would have to be calculated. If the fluid is compressible, an equation of state is needed. In general, p, λ_f, and μ_f are functions of ρ_1.

There are basically two different ways of deriving a constitutive relation for the stress tensor of granular materials — the continuum approach and the statistical approach. We use the continuum approach in our analysis. Goodman and Cowin [34,35] develop a continuum theory for the stress tensor of a granular media. Cowin [36] shows that by including effects due to the gradient of the volume fraction, the Mohr–Coulomb yield criterion for granular materials can be modeled. Savage [37] uses standard representation theorems to derive an expression for $\mathbf{T}_\mathrm{s}$ similar to that of Goodman and Cowin [35]. Savage [38] gives a review of the expressions proposed for $\mathbf{T}_\mathrm{s}$. Massoudi [26], in modeling the solid particles in a fluidized bed, uses an expression similar to that derived by Savage [37]. In this study, we assume that the stress tensor for a granular material is given by Rajagopal and Massoudi [39]

$$\begin{aligned}\mathbf{T}_\mathrm{s} = {} & [\hat{\beta}_0(\rho_2) + \hat{\beta}_1(\rho_2)\mathrm{grad}\,\rho_2 \cdot \mathrm{grad}\,\rho_2 \\ & + \hat{\beta}_2(\rho_2)\mathrm{tr}\,\mathbf{D}_2]\mathbf{I} + \hat{\beta}_3(\rho_2)\mathbf{D}_2 \\ & + \hat{\beta}_4(\rho_2)\mathrm{grad}\,\rho_2 \otimes \mathrm{grad}\,\rho_2,\end{aligned} \tag{17}$$

where $\cdot$ denotes the scalar product of two vectors and $\otimes$ denotes the outer, or tensor, product of two vectors. The spherical part of the stress in Eq. (17) can be interpreted as the solid pressure p_s. The material moduli $\hat{\beta}_1$ and $\hat{\beta}_4$ are material parameters that reflect the distribution of the granular particles, and $\hat{\beta}_0$ plays a role analogous to pressure in a compressible fluid and is given by an equation of state. The material modulus $\hat{\beta}_2$ is a viscosity akin to the second coefficient of viscosity in a compressible fluid and $\hat{\beta}_3$ denotes the viscosity (i.e., the resistance of the material to flow) of the granular solids. Recently, Rajagopal and Massoudi [39] have outlined an experimental/theoretical approach to determine these material moduli. Based on the available experimental measurements of Savage [37], Savage and Sayed [40], and Hanes and Inman [41] and the computer simulations of Walton and Braun [42,43], it is clear that granular materials exhibit normal stress effects. The above model (Eq. (17)) is a simplified version of the model proposed by Rajagopal and Massoudi [39], which predicts the possibility of both the normal stress differences. Furthermore, Boyle and Massoudi [44], using Enskog's dense gas theory, have obtained explicit expressions for the material moduli $\hat{\beta}_0$ through $\hat{\beta}_4$.

There are several interesting and recently published articles which deal with many of the issues of interest in granular materials. One main issue which is not addressed in the above formulation of the stress tensor for the granular materials is the concept of 'yield'. Many researchers in the field, perhaps beginning with Bagnold [45], have observed or have proposed that the stress tensor can be decomposed into two parts: one is for the slow flow with the onset of yield, usually referred to as the frictional flow regime, and the other is for the fast or collisional flow, sometimes referred to as the viscous or the kinetic flow regime (cf. Refs. [46–49]). For a general approach to the flow of granular materials, we refer the reader to the review article by Hutter and Rajagopal [50], and the books by Nedderman [51] and Mehta [52].

A mixture stress tensor is defined as (cf. Ref. [53])

$$\mathbf{T}_{\mathrm{m}} = \mathbf{T}_1 + \mathbf{T}_2, \tag{18}$$

where

$$\mathbf{T}_1 = (1-\nu)\mathbf{T}_{\mathrm{f}} \quad \text{and} \quad \mathbf{T}_2 = \mathbf{T}_{\mathrm{s}}, \tag{19}$$

so that the mixture stress tensor reduces to that of a pure fluid as $\nu \to 0$ and to that of a granular material as $\phi \to 0$. $\mathbf{T}_2$ may also be written as $\mathbf{T}_2 = \nu\hat{\mathbf{T}}_{\mathrm{s}}$, where $\hat{\mathbf{T}}_{\mathrm{s}}$ may be thought of as representing the stress tensor for some (quite densely packed) reference configuration of the granular material.

The mechanical interaction between the mixture components, $\mathbf{f}_{\mathrm{I}}$, is written as [54]

$$\begin{aligned} \mathbf{f}_{\mathrm{I}} = {} & A_1 \operatorname{grad}\nu + A_2\mathscr{F}(\nu)(\mathbf{v}_2 - \mathbf{v}_1) \\ & + A_3\nu(2\operatorname{tr}\mathbf{D}_1^2)^{-1/4}\mathbf{D}_1(\mathbf{v}_2 - \mathbf{v}_1) \\ & + A_4\nu(\mathbf{W}_2 - \mathbf{W}_1)(\mathbf{v}_2 - \mathbf{v}_1) + A_5\mathbf{a}_{\nu m}, \end{aligned} \tag{20}$$

where $\mathbf{a}_{\nu m}$ is a properly frame invariant measure of the relative acceleration between the mixture components and $\mathscr{F}(\nu)$ represents the dependence of the drag coefficient on the volume fraction. The terms in Eq. (20) reflect the presence of density gradients,[1] drag, "slip-shear" lift, "spin" lift, and virtual mass, respectively. Müller's [55] work indicates that a term of the form $A_1 \operatorname{grad}\nu$ must be included in the interactions in order to get well-posed problems. The term multiplying A_3 is a generalization of Saffman's [56,57] single particle result first proposed in this form by McTigue et al. [58].

[1] The actual form of this interaction should include the terms $\alpha_1 \operatorname{grad}\rho_1 + \alpha_2 \operatorname{grad}\rho_2$ where α_1 and α_2 are constants. If we assume that the system is a saturated mixture with incompressible components, this expression simplifies to $A_1 \operatorname{grad}\nu$ where $A_1 = \alpha_2 - \alpha_1$. Since no information concerning the coefficients α_1 and α_2 is available and a term of the same form arises from the granular solid stress tensor, this term will be neglected in the present work.

4. Governing equations

Eqs. (12), (13), (16), (17), and (20) are combined to yield the equations describing the flow of a mixture of a Navier–Stokes fluid and a granular solid. Virtual mass effects are neglected:

$$\begin{aligned} \rho_1\left[\frac{\partial\mathbf{v}_1}{\partial t} + (\operatorname{grad}\mathbf{v}_1)\mathbf{v}_1\right] & \\ = (\operatorname{grad}\nu)p - (1-\nu)\operatorname{grad}p & \\ + \lambda_{\mathrm{f}}[(-\operatorname{grad}\nu)\operatorname{div}\mathbf{v}_1 + (1-\nu)\operatorname{grad}(\operatorname{div}\mathbf{v}_1)] & \\ + (1-\nu)(\operatorname{grad}\lambda_{\mathrm{f}})\operatorname{div}\mathbf{v}_1 & \\ + 2\mu_{\mathrm{f}}[(-\operatorname{grad}\nu)\mathbf{D}_1 + (1-\nu)\operatorname{div}\mathbf{D}_1] & \\ + 2(1-\nu)(\operatorname{grad}\mu_{\mathrm{f}})\mathbf{D}_1 + \rho_1\mathbf{b}_1 & \\ + A_2\mathscr{F}(\nu)(\mathbf{v}_2 - \mathbf{v}_1) & \\ + A_3\nu(2\operatorname{tr}\mathbf{D}_1^2)^{-1/4}\mathbf{D}_1(\mathbf{v}_2 - \mathbf{v}_1) & \\ + A_4\nu(\mathbf{W}_2 - \mathbf{W}_1)(\mathbf{v}_2 - \mathbf{v}_1), & \end{aligned} \tag{21}$$

$$\begin{aligned} \rho_2\left[\frac{\partial\mathbf{v}_2}{\partial t} + (\operatorname{grad}\mathbf{v}_2)\mathbf{v}_2\right] & \\ = \operatorname{grad}\beta_0 + \operatorname{grad}\beta_1(\operatorname{grad}\rho_2\cdot\operatorname{grad}\rho_2) & \\ + 2\beta_1\{\operatorname{grad}(\operatorname{grad}\rho_2)\}^{\mathrm{T}}\operatorname{grad}\rho_2 & \\ + \operatorname{grad}\beta_2(\operatorname{div}\mathbf{v}_2) + \beta_2\operatorname{grad}(\operatorname{div}\mathbf{v}_2) + \beta_3\operatorname{div}\mathbf{D}_2 & \\ + (\operatorname{grad}\beta_3)\mathbf{D}_2 + \beta_4[\{\operatorname{grad}(\operatorname{grad}\rho_2)\}\operatorname{grad}\rho_2 & \\ + (\operatorname{grad}\rho_2)\operatorname{div}(\operatorname{grad}\rho_2)] & \\ + \operatorname{grad}\beta_4[\operatorname{grad}\rho_2\otimes\operatorname{grad}\rho_2] & \\ + \rho_2\mathbf{b}_2 - A_1\operatorname{grad}\nu - A_2\mathscr{F}(\nu)(\mathbf{v}_2 - \mathbf{v}_1) & \\ - A_3\nu(2\operatorname{tr}\mathbf{D}_1^2)^{-1/4}\mathbf{D}_1(\mathbf{v}_2 - \mathbf{v}_1) & \\ - A_4\nu(\mathbf{W}_2 - \mathbf{W}_1)(\mathbf{v}_2 - \mathbf{v}_1). & \end{aligned} \tag{22}$$

Eqs. (21) and (22) are written in dimensionless form as in Ref. [59]

$$\begin{aligned} (1-\nu)\rho_{\mathrm{f}}\left[\frac{\partial\mathbf{V}_1}{\partial\tau} + (\operatorname{grad}\mathbf{V}_1)\mathbf{V}_1\right] & \\ = (\operatorname{grad}\nu)P - (1-\nu)\operatorname{grad}P & \\ + \Lambda_{\mathrm{f}}[(-\operatorname{grad}\nu)\operatorname{div}\mathbf{V}_1 + (1-\nu)\operatorname{grad}(\operatorname{div}\mathbf{V}_1)] & \\ + (1-\nu)(\operatorname{grad}\Lambda_{\mathrm{f}})\operatorname{div}\mathbf{V}_1 & \end{aligned}$$

$$+\frac{2}{\mathrm{Re}}[(-\operatorname{grad} v)\mathbf{D}_1+(1-v)\operatorname{div}\mathbf{D}_1]$$

$$+\ 2(1-v)\left(\operatorname{grad}\frac{1}{\mathrm{Re}}\right)\mathbf{D}_1$$

$$+\frac{\rho_{\mathrm{f}}}{\mathrm{Fr}}(1-v)\mathbf{b}_1+C_2\mathscr{F}(v)(\mathbf{V}_2-\mathbf{V}_1)$$

$$+\ C_3 v(2\operatorname{tr}\mathbf{D}_1^2)^{-1/4}\mathbf{D}_1(\mathbf{V}_2-\mathbf{V}_1)$$

$$+\ C_4 v(\mathbf{W}_2-\mathbf{W}_1)(\mathbf{V}_2-\mathbf{V}_1), \tag{23}$$

$$v\rho_{\mathrm{s}}\left[\frac{\partial\mathbf{V}_2}{\partial t}+(\operatorname{grad}\mathbf{V}_2)\mathbf{V}_2\right]$$

$$=\operatorname{grad}\hat{B}_0+\operatorname{grad}\hat{B}_1(\operatorname{grad}v\cdot\operatorname{grad}v)$$

$$+\ 2\hat{B}_1\{\operatorname{grad}(\operatorname{grad}v)\}^{\mathrm{T}}\operatorname{grad}v+\operatorname{grad}\hat{B}_2(\operatorname{div}\mathbf{V}_2)$$

$$+\ \hat{B}_2\operatorname{grad}(\operatorname{div}\mathbf{V}_2)+\hat{B}_3\operatorname{div}\mathbf{D}_2+(\operatorname{grad}\hat{B}_3)\mathbf{D}_2$$

$$+\ \hat{B}_4[\{\operatorname{grad}(\operatorname{grad}v)\}\operatorname{grad}v$$

$$+(\operatorname{grad}v)\operatorname{div}(\operatorname{grad}v)]$$

$$+\operatorname{grad}\hat{B}_4[\operatorname{grad}v\otimes\operatorname{grad}v]$$

$$+\frac{\rho_{\mathrm{s}}}{\mathrm{Fr}}v\mathbf{b}_2-C_2\mathscr{F}(v)(\mathbf{V}_2-\mathbf{V}_1)$$

$$-\ C_3 v(2\operatorname{tr}\mathbf{D}_1^2)^{-1/4}\mathbf{D}_1(\mathbf{V}_2-\mathbf{V}_1)$$

$$-\ C_4 v(\mathbf{W}_2-\mathbf{W}_1)(\mathbf{V}_2-\mathbf{V}_1), \tag{24}$$

where the following dimensionless groups are identified:

$$\mathrm{Re}=\frac{\rho_0 u_0 L}{\mu_{\mathrm{f}}},\quad \Lambda_{\mathrm{f}}=\frac{\lambda_{\mathrm{f}}}{\rho_0 u_0 L},\quad \mathrm{Fr}=\frac{u_0^2}{Lg}, \tag{25}$$

$$\hat{B}_0=\frac{\beta_0}{\rho_0 u_0^2},\quad \hat{B}_1=\frac{\beta_1}{\rho_0 u_0^2 L^2},\quad \hat{B}_2=\frac{\beta_2}{\rho_0 u_0 L}, \tag{26}$$

$$\hat{B}_3=\frac{\beta_3}{\rho_0 u_0 L},\quad \hat{B}_4=\frac{\beta_4}{\rho_0 u_0^2 L^2},\quad C_1=\frac{A_1}{\rho_0 u_0^2}, \tag{27}$$

$$C_2=\frac{A_2 L}{\rho_0 u_0},\quad C_3=\frac{A_3 L^{1/2}}{\rho_0 u_0^{1/2}},\quad C_4=\frac{A_4}{\rho_0}. \tag{28}$$

5. Couette flow between rotating cylinders

There are many interesting boundary value problems from an analytical point of view, and several important simple problems from an experimental point of view. With the formulation of the problem using the theory of interacting continua, flow between two parallel plates and flow in a pipe were studied by Johnson et al. [59,60]. Later, a modified form of the mixture theory was proposed where the effect of fluid pressure gradient was included in the interaction force (cf. Ref. [61]). Another interesting viscometric flow is that of a mixture of fluid and particles between two rotating cylinders. This flow, for example, can be experimentally used as a means to measure the viscosity of a suspension (cf. Ref. [62]).

Consider the couette flow of a mixture confined to the gap between two infinitely long, concentric, rotating cylinders of circular cross section. Let the wall of the interior cylinder be located at $r=1$ and the outer cylinder at r_0 ($r_0>1$). If the flow is steady and laminar, the velocity profiles and solids distribution can be assumed to have the form

$$\mathbf{V}_1=V(r)\mathbf{e}_\theta,\quad \mathbf{V}_2=U(r)\mathbf{e}_\theta,\quad v=v(r) \tag{29}$$

in a cylindrical polar coordinate system. The balance of mass equations are automatically satisfied by these assumptions. Virtual mass effects and body forces are neglected.

The balance of linear momentum given by Eqs. (23) and (24) are greatly simplified by these assumptions and may be written in component form as

$$(1-v)\rho_{\mathrm{f}}\frac{1}{r}V^2+\frac{\mathrm{d}v}{\mathrm{d}r}P-(1-v)\frac{\partial P}{\partial r}$$
$$+\ \frac{1}{2}C_3 v\left|V'-\frac{1}{r}V\right|^{-1/2}\left(V'-\frac{1}{r}V\right)(U-V)$$
$$+\ \frac{1}{2}C_4 v\left(V'+\frac{1}{r}V-U'-\frac{1}{r}U\right)(U-V)=0, \tag{30}$$

$$(1-v)\left(V''+\frac{1}{r}V'-\frac{1}{r^2}V\right)-v'\left(V'-\frac{1}{r}V\right)$$
$$+\ C_2\,\mathrm{Re}\,\mathscr{F}(v)(U-V)=0, \tag{31}$$

$$B_3\left\{(v+v^2)\left(U''+\frac{1}{r}U'-\frac{1}{r^2}U\right)+(2v+1)v'\right.$$
$$\left.\times(U'-\frac{1}{r}U)\right\}-2C_2\mathscr{F}(v)(U-V)=0, \tag{32}$$

$$v\rho_s\frac{1}{r}U^2 + B_0 v' + (B_1 + B_4)\{2(1+v+v^2)v'v''$$

$$+ (2v+1)(v')^3\} + B_4\frac{1}{r}(v')^2(1+v+v^2)$$

$$-\frac{1}{2}C_3 v\left|V' - \frac{1}{r}V\right|^{-1/2}\left(V' - \frac{1}{r}V\right)(U-V)$$

$$-\frac{1}{2}C_4 v\left(V' + \frac{1}{r}V - U' - \frac{1}{r}U\right)(U-V) = 0, \tag{33}$$

where the prime denotes a derivative with respect to r.

Consider Eqs. (31)–(33) in the limiting case as v approaches 0. Note that it is assumed, based on physical considerations, that $\mathscr{F}(v) \to 0$ as $v \to 0$. If $v \equiv 0$, then, Eqs. (32) and (33) vanish. Eq. (31) becomes

$$V'' + \frac{1}{r}V' - \frac{1}{r^2}V = 0, \tag{34}$$

which is consistent with couette flow of pure fluid [70].

Eqs. (31)–(33) have to be solved numerically. In order to do so, the following definitions are introduced:

$$\mathscr{D}_1 = C_2\,\mathrm{Re}, \quad \mathscr{D}_2 = \frac{2C_2}{B_3}, \quad \mathscr{B}_1 = \frac{B_0}{B_1 + B_4},$$

$$\mathscr{P} = \frac{\rho_s}{B_1 + B_4}, \tag{35}$$

$$\mathscr{B}_2 = \frac{B_4}{B_1 + B_4}, \quad \mathscr{L}_1 = \frac{-C_3}{2(B_1 + B_4)},$$

$$\mathscr{L}_2 = \frac{-C_4}{2(B_1 + B_4)}. \tag{36}$$

Though introduced primarily for numerical convenience, these dimensionless parameters have physical interpretations. $\mathscr{D}_1$ is the ratio of the drag force exerted on the fluid by the solid to the viscous forces within the fluid phase. $\mathscr{D}_2$ is the ratio of the drag force exerted on the solid by the fluid to the viscous forces within the solid component. $\mathscr{B}_1$ and $\mathscr{B}_2$ are combinations of the material coefficients from the granular solid constitutive equation; they affect the distribution of solid in the flow but do not *directly* influence the velocity profiles of either fluid or solid. $\mathscr{L}_1$ incorporates the effect of slip-shear lift into the equations; $\mathscr{L}_2$ is the spin lift coefficient. The fluid and solid velocity fields exert an effect on the distribution of the granular solid only when $\mathscr{L}_1 \neq 0$ and/or $\mathscr{L}_2 \neq 0$. The parameter $\mathscr{P}$ is the ratio of the centrifugal forces to the forces generated by solid–solid interactions within the granular material. With these definitions, Eqs. (31)–(33) are rewritten as

$$(1-v)\left(V'' + \frac{1}{r}V' - \frac{1}{r^2}V\right) - v'\left(V' - \frac{1}{r}V\right)$$

$$+ \mathscr{D}_1\mathscr{F}(v)(U-V) = 0, \tag{37}$$

$$(v+v^2)\left(U'' + \frac{1}{r}U' - \frac{1}{r^2}U\right) + (2v+1)v'$$

$$\times\left(U' - \frac{1}{r}U\right) - \mathscr{D}_2\mathscr{F}(v)(U-V) = 0, \tag{38}$$

$$\mathscr{P}v\frac{1}{r}U^2 + \mathscr{B}_1 v' + 2(1+v+v^2)v'v'' + (2v+1)(v')^3$$

$$+ \mathscr{B}_2\frac{1}{r}(v')^2(1+v+v^2) - \mathscr{L}_1 v\left|V' - \frac{1}{r}V\right|^{-1/2}$$

$$\times\left(V' - \frac{1}{r}V\right)(U-V) - \mathscr{L}_2 v$$

$$\times\left(V' + \frac{1}{r}V - U' - \frac{1}{r}U\right)(U-V) = 0. \tag{39}$$

Note that when $\mathscr{L}_1 = \mathscr{L}_2 = 0$, Eq. (39) is still coupled to Eqs. (37) and (38) through the velocity field. Based on Batchelor's [63] result for sedimentation of a dilute suspension of spheres, Drew [64] proposes the following approximation for $\mathscr{F}(v)$:

$$\mathscr{F}(v) = v(1 + 6.55v). \tag{40}$$

This expression is used here with the understanding that it is not valid as $v \to v_m$, where v_m is the maximum packing fraction of the granular material.

5.1. Boundary conditions

Within the context of mixture theory, boundary conditions impose an added difficulty. In general, boundary conditions can have a mathematical or a physical origin. Whether we specify the displacement vector, or the velocity vector, or velocity gradient, or the traction, or the stress tensor of a given phase or of the mixture, would depend on the given problem. Many times the choice is clear, however, in case of a mixture with free surface, or a mixture where slip at the boundary is possible, the choice is not clear. Furthermore, in cases where a constitutive relation is used to describe a non-linear material, new and additional boundary conditions may arise due to the higher order terms in the constituive relation. These issues have limited the application of mixture theory to problems of practical interest. It is only in the last decade or so that Rajagopal and co-workers (cf. Ref. [28]) have attempted to solve a series of interesting problems.

A look at Eqs. (30)–(33) would reveal that two boundary conditions are necessary for each velocity component, two boundary condition for the volume fraction of the solid particles, and one boundary condition for the fluid pressure. In general, the fluid pressure is eliminated through cross-differentiation; this however, increases the order of the equations, and as a result more boundary conditions are needed. In many cases, physical boundary conditions such as the flow rate of the mixture, or the volumetric flow rate, or... can also be used (cf. Ref. [65]). For a recent review of the issues of interest in boundary effects in granular materials, the reader is referred to Zheng and Hill [66]. For the mixture theory, as presented here, the details of the selection and appropriateness of the boundary conditions are given in Refs. [59,60,68].

Adherence boundary conditions are imposed on both the constituents at both cylinder walls:

$$U(1) = V(1) = W_{\mathrm{i}}, \tag{41}$$

$$U(R_0) = V(R_0) = W_{\mathrm{o}}, \tag{42}$$

where W_{i} is the velocity of the inner cylinder and W_{o} is the velocity of the outer cylinder.

The appropriate conditions on v are the boundary value of v at one cylinder wall and a prescribed average volume fraction, defined as

$$N = 2\pi \int_1^{R_0} vr\,\mathrm{d}r. \tag{43}$$

5.2. Results

Eqs. (31)–(33) are solved using the collocation code COLSYS [67]. COLSYS was chosen for this problem after comparison with multiple shooting and finite difference codes. The collocation method proved superior to the other methods in both the ability to calculate solutions and the computer time required to calculate them. Solutions obtained using different methods matched each other closely in cases where more than one method was effective.

Collocation is implemented by COLSYS using B-spline basis functions. The error in the calculated solutions is estimated by mesh halving and checked against user-prescribed tolerances. The mesh points are then automatically redistributed to roughly equalize the error in each subinterval. Error tolerances on the volume fraction, solid velocity, fluid velocity, and their derivatives were specified as 10^{-5} for the solutions given below. The integral condition on the volume fraction was implemented using a secant shooting method to refine initial guesses for the value of v at the pipe wall.

There are some restrictions on the values the parameters may take. Clearly, the drag coefficient, Reynold's number, and material viscosities must be greater than zero and therefore $\mathscr{D}_1$ and $\mathscr{D}_2$ must also be greater than zero. Similarly, the lift coefficients $\mathscr{L}_1$ and $\mathscr{L}_2$ must be less than zero for the direction of lift to match experimental observations [69]. Based on results obtained for flow between parallel plates [59] it is assumed that $B_1 + B_4 < 0$. Since B_0 is also less than zero, $\mathscr{B}_1$ is greater than zero. $\mathscr{B}_2$ is also assumed to be greater than zero. The parameter $\mathscr{P}$ must always be less than zero. In summary, $\mathscr{D}_1, \mathscr{D}_2, \mathscr{B}_1$, and $\mathscr{B}_2$ are all greater than zero; $\mathscr{L}_1, \mathscr{L}_2$, and $\mathscr{P}$ are less than zero.

A few representative velocity and volume fraction profiles are shown here, with a discussion of the effects of the parameters. Though any combination of velocities may be specified for the two

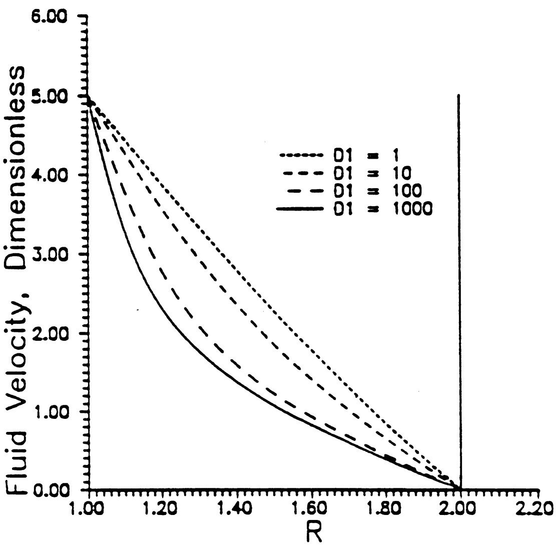

Fig. 1. Couette flow with rotating inner cylinder: effect of $\mathscr{D}_1$ on fluid velocity profile. $\mathscr{D}_2 = 10$, $\mathscr{B}_1 = 4$, $\mathscr{B}_2 = 4$, $\mathscr{L}_1 = \mathscr{L}_2 = 0$, $\mathscr{P} = -1$, and $N = 0.4$.

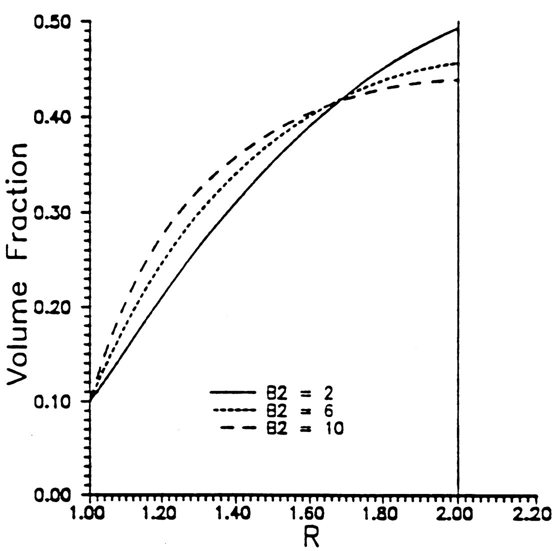

Fig. 2. Couette flow with rotating inner cylinder: effect of $\mathscr{B}_2$ on volume fraction, $\mathscr{B}_1 = 1$, $\mathscr{D}_1 = 10$, $\mathscr{D}_2 = 1$, $\mathscr{L}_1 = \mathscr{L}_2 = 0$, $\mathscr{P} = -1$, and $N = 0.4$.

cylinders, for convenience only two cases are considered. Figs. 1–4 are obtained with $W_i = 5$ and $W_o = 0$. Figs. 5–8 are the corresponding results for $W_i = 0$ and $W_o = 5$.

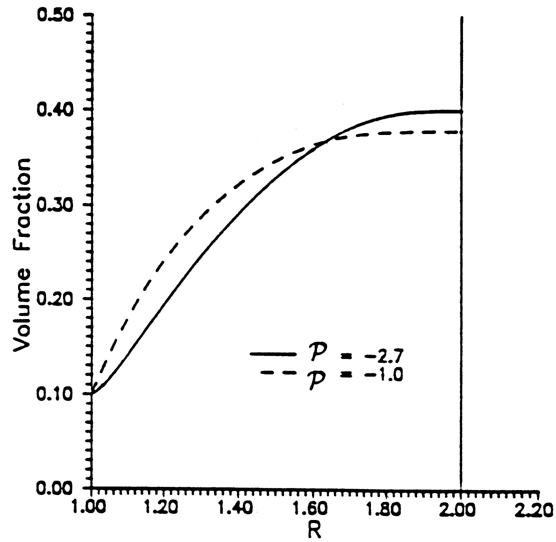

Fig. 3. Couette flow with rotating inner cylinder: effect of $\mathscr{P}$ on volume fraction. $\mathscr{D}_1 = 10$, $\mathscr{D}_2 = 5$, $\mathscr{B}_1 = 4$, $\mathscr{B}_2 = 4$, $\mathscr{L}_1 = \mathscr{L}_2 = 0$, and $N = 0.3538$.

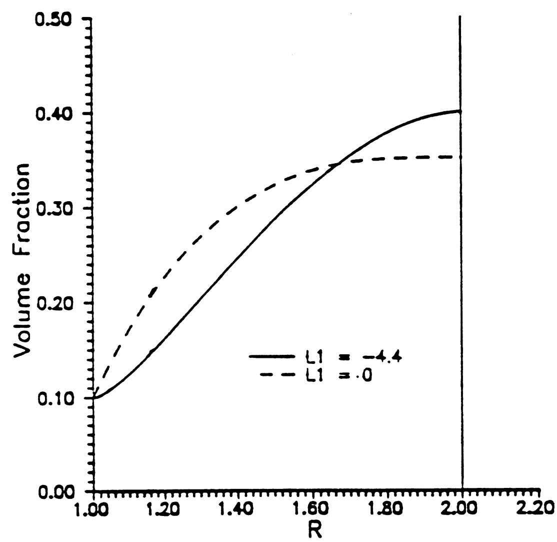

Fig. 4. Couette flow with rotating inner cylinder: effect of lift on volume fraction. $\mathscr{L}_2 = 0$, $N = 0.3294$, $\mathscr{D}_1 = 10$, $\mathscr{D}_2 = 5$, $\mathscr{B}_1 = 4$, and $\mathscr{B}_2 = 4$, and $\mathscr{P} = -1$.

The parameter $\mathscr{D}_1$ affects the fluid velocity profile as shown in Fig. 1. Increasing values of $\mathscr{D}_1$ results in a decrease in the fluid velocity. The solid velocity also decreases due to the coupling of the two equations. Physically, an increase in the value of $\mathscr{D}_1$ corresponds to either an increase in the drag

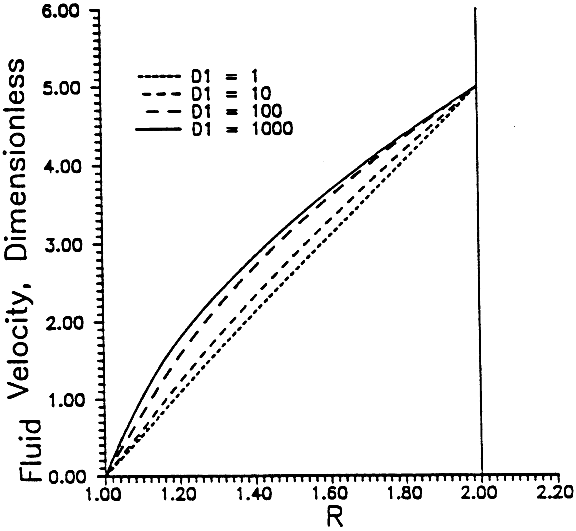

Fig. 5. Couette flow with rotating outer cylinder: effect of $\mathscr{D}_1$ on fluid velocity profile. $\mathscr{D}_2 = 10$, $\mathscr{B}_1 = 4$, $\mathscr{B}_2 = 4$, $\mathscr{L}_1 = \mathscr{L}_2 = 0$, $\mathscr{P} = -1$, and $N = 0.4$.

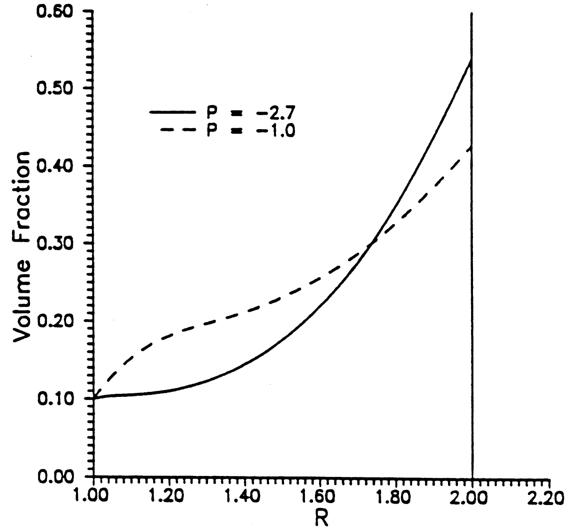

Fig. 7. Couette flow with rotating outer cylinder: effect of $\mathscr{P}$ on volume fraction. $\mathscr{D}_1 = 10$, $\mathscr{D}_2 = 5$, $\mathscr{B}_1 = 4$, $\mathscr{B}_2 = 4$, $\mathscr{L}_1 = \mathscr{L}_2 = 0$, and $N = 0.2955$.

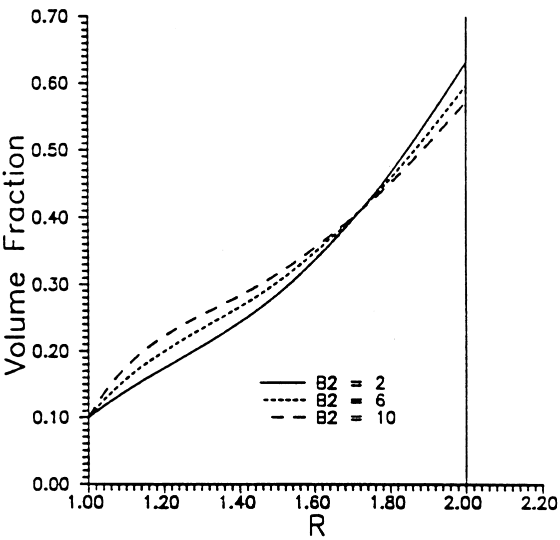

Fig. 6. Couette flow with rotating outer cylinder: effect of $\mathscr{B}_2$ on volume fraction. $\mathscr{B}_1 = 1$, $\mathscr{D}_1 = 10$, $\mathscr{D}_2 = 1$, $\mathscr{L}_1 = \mathscr{L}_2 = 0$, $\mathscr{P} = -1$, and $N = 0.4$.

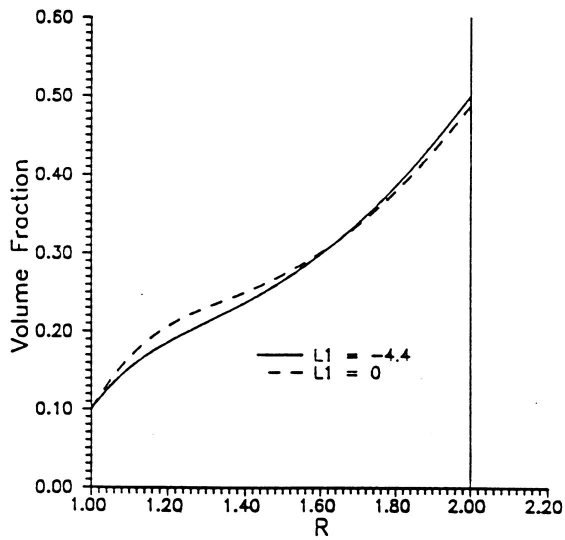

Fig. 8. Couette flow with rotating outer cylinder: effect of lift on volume fraction. $\mathscr{L}_2 = 0$, $N = 0.3422$, $\mathscr{D}_1 = 10$, $\mathscr{D}_2 = 5$, $\mathscr{B}_1 = 4$, $\mathscr{B}_2 = 4$, and $\mathscr{P} = -1$.

coefficient or an increase in the Reynolds number or both.

Increasing $\mathscr{D}_2$ (not shown) results in an increase in fluid velocity and a corresponding increase in solid velocity. Physically, an increase in the value of $\mathscr{D}_2$ corresponds to either an increase in the drag coefficient or a decrease in the non-dimensional number B_3.

Fig. 2 shows plots of volume fraction versus r for some representative values of $\mathscr{B}_1$ and $\mathscr{B}_2$ (with

$\mathscr{L}_1 = \mathscr{L}_2 = 0$). Note that an integral condition (cf. Eq. (43)) is used and the areas under each of the curves are the same, although the boundary condition at the outer tube wall is different for each curve.

Fig. 3 shows the effect of $\mathscr{P}$ on volume fraction. A larger value of $\mathscr{P}$ results in a greater portion of the granular solid collecting near the outer cylinder wall (see Fig. 3). Physically, an increase in the magnitude of $\mathscr{P}$ corresponds to an increase in ρ_s, the density of the granular solid. The change in solid distribution has relatively little effect on the velocity profiles.

Fig. 4 shows a representative result from a parametric study of the effects lift on the volume fraction. Spin lift and slip-shear lift appear to have essentially the same effect on the flow. Increasing $\mathscr{L}_1$ and/or $\mathscr{L}_2$, and thus the significance of lift, causes the granular solid to move away from the inner wall and collect towards the outer wall. Again, the changes in volume fraction due to changing the lift coefficient have relatively little effect on the velocity profiles.

Figs. 5–8 show result analogous to those discussed above for a stationary inner cylinder and a rotating outer cylinder. The parameter $\mathscr{D}_1$ affects the fluid velocity profile as shown in Fig. 5. Increasing values of $\mathscr{D}_1$ results in an increase in the fluid velocity. The solid velocity also increases due to the coupling of the two equations. Note that these results are opposite to those in Fig. 1 for a rotating inner cylinder.

Increasing $\mathscr{D}_2$ results in a decrease in fluid velocity and a corresponding decrease in solid velocity. Physically, an increase in the value of $\mathscr{D}_2$ corresponds to either an increase in the drag coefficient or a decrease in the non-dimensional number B_3. Note again the reversal of the effects for the rotating outer cylinder as compared to the rotating inner cylinder. The solid leads the fluid when the outer cylinder is rotating; the fluid leads the solid when the inner cylinder is rotating.

Fig. 6 shows plots of volume fraction versus r for some representative values of $\mathscr{B}_1$ and $\mathscr{B}_2$ (with $\mathscr{L}_1 = \mathscr{L}_2 = 0$). Note that an integral condition (cf. Eq. (43)) is used and the areas under each of the curves are the same, although the boundary condition at the outer tube wall is different for each curve. Though $\mathscr{B}_1$ and $\mathscr{B}_2$ appear to cause the same relative changes in the volume fraction curves regardless of which cylinder is rotating, the overall shape of the curves are different in Fig. 6 versus Fig. 2.

Fig. 7 shows the effect of $\mathscr{P}$ on the volume fraction. A larger value of $\mathscr{P}$ results in a greater portion of the granular solid collecting near the outer cylinder wall (see Fig. 7). Physically, an increase in the magnitude of $\mathscr{P}$ corresponds to an increase in ρ_s, the density of the granular solid. The change in solid distribution has relatively little effect on the velocity profiles. The change in the volume fraction profile is more pronounced when the outer cylinder is rotating than when the inner cylinder is rotating (Fig. 3).

Fig. 8 shows a representative result from a parametric study of the effects lift on the volume fraction. Increasing $\mathscr{L}_1$ and/or $\mathscr{L}_2$, and thus the significance of lift, causes the granular solid to move away from the inner wall and collect towards the outer wall. This effect appears much less significant for a rotating outer cylinder than for a rotating inner cylinder (see Fig. 4). Again, the changes in volume fraction due to changes in the lift coefficient have relatively little effect on the velocity profiles.

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